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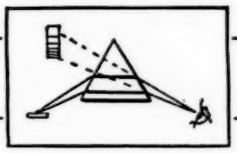

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The Aberration of Clocks and the Clock Paradox

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(Received July 7, 1937)

THE ABERRATION OF CLOCKS

IN a preceding paper,[1] I have shown that clocks experiencing the Larmor-Lorentz frequency change are altered in setting by transport by exactly the amount necessary to make the measured times of transit of light signals invariant on a uniformly moving body, subject to the Fitzgerald contraction. Study of this change of setting suggests that if clocks are transported over the same measured distance at different times, when the velocity of the observing platform is different, their settings should differ.

From the preceding paper, I take the expression for the change in setting of a clock moved between two points at the distance L_s apart (when the observing platform is stationary) at the observed velocity W_0, the velocity of the observing platform through the ether being v. It is

$$\Delta S=\frac{(L_s/c)(1-v^2/c^2)^{\frac{1}{2}}}{(W_0/c)(1-v^2/c^2)}\left[(1-v^2/c^2)^{\frac{1}{2}}-\left\{1-\frac{\left(v+\dfrac{W_0(1-v^2/c^2)}{W_0v/c^2+(1+W_0^2/c^2)^{\frac{1}{2}}}\right)^2}{c^2}\right\}^{\frac{1}{2}}\right]. \tag{1}$$
$$W_0v/c^2+(1+W_0^2/c^2)^{\frac{1}{2}}$$

To solve this, put

$$v/c=x, \tag{2}$$

$$W_0/c=\alpha, \tag{3}$$

$$z=\frac{\alpha(1-x^2)}{x\alpha+(1+\alpha^2)^{\frac{1}{2}}}. \tag{4}$$

Eq. (1) then becomes

$$\Delta S=\frac{(L_s/c)(1-x^2)^{\frac{1}{2}}}{z}[(1-x^2)^{\frac{1}{2}}-(1-(x+z)^2)^{\frac{1}{2}}]; \tag{5}$$

then

$$x+z=\frac{\alpha+x(1+\alpha^2)^{\frac{1}{2}}}{x\alpha+(1+\alpha^2)^{\frac{1}{2}}},$$

$$1-(x+z)^2=\frac{1-x^2}{[x\alpha+(1+\alpha^2)^{\frac{1}{2}}]^2} \tag{6}$$

[1] "Light Signals on Moving Bodies as Measured by Transported Rods and Clocks," J. O. S. A. **27**, 263 (1937).

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and

$$(1-(x+z)^2)^{\frac{1}{2}}=\frac{(1-x^2)^{\frac{1}{2}}}{x\alpha+(1+\alpha^2)^{\frac{1}{2}}},$$

so that

$$\Delta S=\frac{L_s}{c}\frac{(1-x^2)}{\alpha-(1x^2)}[(x\alpha+(1+\alpha^2)^{\frac{1}{2}}]\left(1-\frac{1}{x\alpha+(1+\alpha^2)^{\frac{1}{2}}}\right)$$

$$=\frac{L_s}{c\alpha}(x\alpha+(1+\alpha^2)^{\frac{1}{2}}-1)$$

$$=\frac{L_s}{c}\left[x-\frac{1-(1+\alpha^2)^{\frac{1}{2}}}{\alpha}\right] \tag{7}$$

$$=\frac{L_s}{c}\left[\frac{v}{c}-\frac{1-(1+W_0^2/c^2)^{\frac{1}{2}}}{W_0/c}\right].$$

Suppose that the observing platform is moving with the velocity v_1 at one time, and later is moving with the velocity v_2. Let us at some time during the first period move a clock between two fixed points on the platform, whose distance apart, for the platform stationary, is L_s. If we measure this distance by rods placed successively end to end (i.e., moving infinitely slowly with respect to the platform) we simply measure the distance as L_s, its true value being $L_s(1-v_1^2/c^2)^{\frac{1}{2}}$. According to (7) the setting of the moved clock is changed by

$$\Delta S_1=\frac{L_s}{c}\left[\frac{v_1}{c}-\frac{1-(1+W_0^2/c^2)^{\frac{1}{2}}}{W_0/c}\right]. \tag{8}$$

Let us now, at some time during the second period, move another clock (which may be the clock left at the origin) to the distant point. Its change of setting, by (7) is

$$\Delta S_2=\frac{L_s}{c}\left[\frac{v_2}{c}-\frac{1-(1+W_0^2/c^2)^{\frac{1}{2}}}{W_0/c}\right]. \tag{9}$$

The two transported clocks will then differ in setting by

$$\Delta S_2-\Delta S_1=\frac{L_s}{c}\left[\frac{v_2-v_1}{c}\right]. \tag{10}$$

If we put T for the time taken by a light signal to traverse the path when the platform is stationary we have

$$\Delta S_2-\Delta S_1=T\left[\frac{v_2-v_1}{c}\right]. \tag{11}$$

Let us now compare this operation with the familiar phenomenon of the aberration of light, as disclosed by observations on a star made with a telescope. In that case we have that the deviation of the star image between epochs is

$$\Delta\sigma=F\left[\frac{v_2-v_1}{c}\right], \tag{12}$$

where F is the focal length of the telescope. We therefore have expressions of similar form for the different positions of a stellar image and the different settings of transported clocks, the one in terms of the *distance* traversed by a light signal, the other in terms of the *time* of transit of a light signal, both measures of the change of velocity of the observing platform with respect to the ether.

It is assumed in the derivation of (11) that the rate of the clock originally transported continuously follows the changing velocity of the platform, without experiencing a change of setting. The factor which gives the clock rate, $(1-v^2/c^2)^{\frac{1}{2}}$, is thus imagined as describing an effect of the nature of a frictional force holding at all instants. In any case the change of setting of clocks at the origin and at the distant point in the experiment as outlined would be the same, since both experience the same change of velocity.

If the velocity of transport of the clocks is made infinitesimal, the change of clock setting approximates the change of transit time of light signals, due to the alteration in the velocity of the platform. In this case the aberration of clocks would exhibit itself as a discrepancy between the times indicated by a continuously running clock at a distant point, and time settings derived from light signals assigned the velocity c.

The clock phenomenon above discussed would be complicated, in the case of the earth, by the diurnal rotation, which would be periodically adding to and subtracting from the velocities of the clocks due to the earth's orbital motion. In order to meet the condition that clocks at both ends of the path should experience the same

changes of velocity, the experiment would have to be performed at the same *sidereal* time, at different epochs. Practically, the effect would be beyond the possibility of detection by any clocks now available. The difference of setting due to the reversal of direction of the earth's orbital motion at epochs six months apart is a fraction of the order of 10^{-4} of the time required for a light signal to traverse the path on the earth, which would in representative experimental conditions be of the order of 10^{-4} second. This would require the originally transported clock to maintain its time for six months to better than 10^{-8} second.

As stated above, this aberration of clocks gives a measure of the change of velocity of the earth with respect to the ether, similar to the aberration of light exhibited by the displacement of a star image in the telescope. It differs from the aberration of light in that it could, theoretically, be detected by observations confined solely to the earth, provided the path of clock transportation were constant in direction, as fixed by a Foucault pendulum or gyroscope. It does not require an energy pattern such as that set up by the stars and observed in a telescope in order to render the shift of the ether with respect to the earth detectable. On the other hand, it does not give the overwhelming probability that it is the earth rather than the ether which moves annually, which is furnished by stellar aberration due to the vast number of stars, as discussed in the previous paper.

The Clock Paradox

In the preceding paper[1] it was shown that the expressions describing the transit of a light signal simultaneously over several uniformly moving bodies could be formulated entirely in terms of the observed relative velocities of the bodies (and the velocity of light). From this result it is tempting to conclude that only relative velocities of material bodies are significant (or existent), and that the occurrence of similar relative motions must always be accompanied by completely similar phenomena.

A hypothetical experiment based on this conclusion leads to the "clock paradox." Given two similar clocks A and B; let B be moved to a distant point at the velocity V with respect to A, and back to A. It will, because of its motion, go at a slower rate, a function of V^2/C^2, and will on its return be *slow with respect to* A. But if relative motion only is of significance, A is likewise moving at the velocity V with respect to B, and hence will, when the clocks are together again, *be slow with respect to* B. Obviously two clocks, side by side, cannot each be slow with respect to the other. There is here a logical inconsistency which demands an examination of the premises.

I propose to discuss this problem from the standpoint of the preceding paper, in which the ether is assumed as the framework with respect to which the alteration of clock rates takes place. It will be shown that, so studied, the behavior of clocks is entirely rational; the "clock paradox" does not arise. As in the case of light signals, velocities with respect to the ether do not appear in the final formulae; but a given relative velocity of two clocks, separated and united, is entirely compatible with one changing its setting, and the other not. In other words, the clock phenomena, unlike the light signal phenomena discussed, are *not* unique functions of the observed relative velocities.

Let us for simplicity first consider that one of our clocks, A, is moving uniformly through the ether, and that the other clock, B, is moved away from A, and back again at a uniform observed velocity. We shall, as before, look upon the change of clock rate, due to motion, as similar to a frictional effect; the operation of reversing the direction of motion introducing of itself no change of setting. Our problem is to find the total change of setting of B due to its motion out and back.

The appropriate expressions, of the form given by Eq. (4) of the previous paper, are

$$\Delta S=\frac{L_0(1-v^2/c^2)^{\frac{1}{2}}}{W_1}\times[(1-v^2/c^2)^{\frac{1}{2}}-(1-(v+W_1)^2/c^2)^{\frac{1}{2}}] \quad (13)$$

and

$$\Delta S'=\frac{L_0(1-v^2/c^2)^{\frac{1}{2}}}{W_2}\times[(1-v^2/c^2)^{\frac{1}{2}}-(1-(v-W_2)^2/c^2)^{\frac{1}{2}}], \quad (14)$$

where W_1 and W_2 are the velocities of clock B

with respect to A on its journeys out and back. The solution of (13) in terms of an observed velocity W_0 has already been obtained in (7). To solve (14) we derive, by exactly similar processes to those already given,

$$W_2=\frac{W_0(1-v^2/c^2)}{(1+W_0^2/c^2)^{\frac{1}{2}}-W_0v/c^2} \tag{15}$$

and finally

$$\Delta S'=\frac{L_o}{c}\left[-\frac{v}{c}-\frac{1-(1+W_0^2/c^2)^{\frac{1}{2}}}{W_0/c}\right]. \tag{16}$$

The sum of ΔS [from (7)] and $\Delta S'$ [from (14)], is the amount by which B is set back with respect to A, by its journey; it is

$$\Delta S+\Delta S'=-\frac{2L_o}{c}\left[\frac{1-(1+W_0^2/c^2)^{\frac{1}{2}}}{W_0/c}\right]. \tag{17}$$

(For small values of W_0/c, this reduces to $\frac{1}{2}T(W_0^2/c^2)$, where $T=2L_o/W_0$.) We learn from this equation that the change of setting of B is independent of v, that is, it is the same whether A is moving uniformly or is stationary with respect to the ether, and that the change of setting of B is expressible in terms of the observed relative velocity of the two clocks.

Let us now take the case where clock B is moving uniformly through the ether, while clock A is moved out and back in the opposite direction to the previous excursion of B. Inspection of the Eqs. (13) to (17) shows that for this second case we obtain *an identical expression* (17) for the amount by which A is set back with respect to B. As a third case consider *both* clocks moved out and back with respect to their uniformly moving locus of coincidence, at velocities such that their relative measured velocity is again W_0. It is clearly possible for their velocities to be such that they each experience the *same* change of setting, with the result that there is *no difference of setting* between the clocks.

It is clear from consideration of these special cases that it is not possible for observers on A and B to predict from their observations of relative velocity in what relation their clocks will stand on returning to coincidence. There exists a series of possible setting differences all compatible with the same observed relative velocities. Eq. (17) gives the *maximum* change of setting which may occur for an observed relative velocity W_0; the actual difference, not predictable from the observed relative velocities alone, may vary between this maximum and zero. When a difference of setting actually occurs that clock of two which is fast, after their separation and reunion, is the one which has moved least with respect to the motion of their point of coincidence through the ether.

In the actual performance of an experiment on these changes of clock settings a common material platform would, for practical reasons, be provided, with reference to which the two clocks would be moved. Our point of intersection of clock motions would then be a point on this platform. What our study has shown is that the differences in clock indications, after the clocks have been moved in various ways with respect to this reference point, are given by Eq. (17) and the similar equations called for by the discussion immediately following. In these equations the state of rest or uniform motion of the reference platform drops out, so that the results are functions only of the relative motions of the clocks *with reference to the uniformly moving platform*. In this sense only can we say that relative motions of matter are alone significant in this problem. The idea that the relative motions of the clocks, *with reference to each other*, are sufficient to predict the resultant conditions, is fallacious. The clock paradox originates in this fallacious conclusion.

The behavior of separated and reunited clocks is very clearly comprehended when we correlate it with the aberration of light. Suppose that observers on each clock observe the variation of position of the stars in the sky during the interval of relative motion of the clocks. If no stellar aberration is observed from one clock, that is the clock whose setting will be unchanged. If the stars are seen from one clock to exhibit a to and fro motion, then that clock will be found to be set back when the clocks are again side by side. If aberration is observed from both clocks, their final difference of setting will be a function of the difference in the observed oscillations of stellar position. In short, in order to predict what changes in setting will occur in clocks moved apart and then together, we must observe their

motions with respect to the ether (locus of stellar light patterns)[2] we cannot determine these changes by observations on their motion with respect to each other.

We thus find that the study of moved clocks, using the Bradley-Fresnel ether, together with the Fitzgerald-Larmor-Lorentz contractions of length and frequency, indicates an entirely consistent and reasonable behavior. The "clock paradox" is a consequence of a sweeping and unqualified application of the hypothesis that relative motion of matter is the only operative factor. This hypothesis is shown by the above study not to be a universal consequence of the length and frequency contractions postulated to explain the Michelson-Morley experiment. That experiment requires processes to account for invariance with respect to uniform motion through the ether; these are furnished by the contractions. There are no relative motions of matter in the Michelson-Morley experiment, and no hypothesis with regard to relative motion of matter is required or involved. Our study shows that, on the basis of the length and frequency contractions, the moved clocks have the same property as the Michelson-Morley apparatus, namely invariance of behavior with respect to the state of rest or motion of *the whole system of clocks* through the ether. The supplementary hypothesis, suggested by extrapolation from the behavior of light signals, that relative motions are alone significant in all cases, is discredited by the physical and logical absurdity of the "clock paradox." The use of the ether as the reference frame is supported by its ability to give a rational account of clock behavior.[3]

[2] The state of rest or uniform motion of the material platform of the preceding paragraph must in the last analysis be determined by recourse to observations on aberration, so that the findings of that paragraph are included in this conclusion.

[3] Another case where confusion results from a loose application of the idea that relative motion of matter is alone significant, is the problem of the aberration of light from binary stars. Spectroscopic binaries, whose components have orbital velocities of the order of magnitude of the earth's, would, if the relative velocities of earth and star were the sole determining factor, necessarily be telescopic binaries with separations of the order of the aberration constant (20″). Such spectroscopic binaries exist, which are not telescopically resolvable. When aberration is ascribed to the *observer's* motion through the ether no such difficulty arises.

Apparent Lengths and Times in Systems Experiencing the Fitzgerald-Larmor-Lorentz Contractions

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(Received July 7, 1937)

IN a previous paper[1] I have discussed the phenomena of light signals on moving bodies, when the moving bodies are subject to a contraction of length in the ratio $(1-v^2/c^2)^{\frac{1}{2}}:1$, and when the clocks used are changed in frequency in the same ratio; v being the velocity of the body, and c the velocity of light, both referred to the luminiferous ether. In the present paper I consider, on the same assumptions,—which derive from the phenomena of the aberration of light, and the Michelson-Morley experiment—measurements of lengths and times on relatively moving bodies, in which light signals are not involved. The formulae to be used are in large part already available from the preceding paper, but are here considered from a different aspect.

We shall use in this study an *observation platform*, an *observed body* and an *observed clock*, all in relative uniform motion with respect to each other. The observation platform is distinguished from the observed body only by the fact that we have selected on it two separated observation points, a and b, from which we observe the other body and the clock. The distance apart of these points we determine with a measuring rod, moved past them at some definite observed velocity; the times at these points we establish by similar clocks which have been synchronized and set together when in coincidence and have been moved apart at some definite observed velocity. Our general problem is to find the relationship between length and time intervals on the observation platform and on the observed body and clock, as functions of their relative velocity and the velocities of the rods and clocks used to locate the points a and b on the observation platform. Our general procedure is to use the method of coincidences,—since we are now excluding light signals. We shall make "simultaneous" observations of the points opposite a and b on the observed body, and we shall observe the times of transit of the observed clock past a and b, both by the clocks at a and b, and by the observed clock itself.

The scheme of labeling the various bodies concerned in the precise way just indicated, is adopted in order to emphasize the real distinction between the bodies, which is obscured by the frequent use in discussions of this problem of the term "stationary" to describe the observation platform, and the term "moving" to describe the observed body and clock. The significant distinction is not in terms of rest or motion, since all the bodies are in relative motion with respect to each other, but lies in the nature and point of origin of the *operations* which are performed.

We shall consider first the times of transit of the observed clock between a and b. Let us designate the distance ab by $L_s(1-v^2/c^2)^{\frac{1}{2}}$, where, as in the preceding paper, L_s is the distance ab when the observation platform is stationary in the ether and v is the velocity of the platform through the ether. Let V be the velocity of the clock with respect to the observing platform, its total velocity with respect to the ether being $v+V$. We then have, by Eq. (1) of the previous paper, that the true time of transit is

$$T=\frac{L_s(1-v^2/c^2)^{\frac{1}{2}}}{V}. \tag{1}$$

The time by the observed clock's own indication is

$$T_0=\frac{L_s(1-v^2/c^2)^{\frac{1}{2}}}{V}\left(1-\frac{(v+V)^2}{c^2}\right)^{\frac{1}{2}}. \tag{2}$$

This we must now put in terms of V_0, the velocity of the clock as observed, as discussed in the preceding paper.

Following the procedure there given, we have

$$V=\frac{V_0(1-v^2/c^2)}{(V_0v/c^2)+(1+V_0^2/c^2)^{\frac{1}{2}}}. \tag{3}$$

[1] "Light Signals on Moving Bodies as Measured by Transported Rods and Clocks," J. O. S. A. **27**, 263 (1937).

In order to solve (2) we then put $v/c=x$ and $V/c=z$, giving

$$T_0=\frac{(L_s/c)(1-x^2)^{\frac{1}{2}}}{z}\left(1-\frac{(x+z)^2}{c^2}\right)^{\frac{1}{2}}. \quad (4)$$

Putting $V_0/c=\alpha$, we have (from appendix to previous paper)

$$z=\frac{\alpha(1-x^2)}{xa+(1+x^2)^{\frac{1}{2}}} \quad (5)$$

and

$$\left(1-\frac{(x+z)^2}{c^2}\right)^{\frac{1}{2}}=\frac{(1-x^2)^{\frac{1}{2}}}{x\alpha+(1+\alpha^2)^{\frac{1}{2}}} \quad (6)$$

leading to

$$T_0=\frac{(L_s/c)(1-x^2)^{\frac{1}{2}}}{\alpha(1-x^2)/(x\alpha+(1+\alpha^2)^{\frac{1}{2}})}\,\frac{(1-x^2)^{\frac{1}{2}}}{x\alpha+(1+x^2)^{\frac{1}{2}}}$$

$$=\frac{L_s}{cx}=\frac{L_s}{V_0} \quad (7)$$

for the time indicated by the observed clock for its transit between a and b.

We now investigate the transit time as recorded by the two clocks at a and b on the observing platform. It will be the product of the true time $(L_s/V)(1-v^2/c^2)^{\frac{1}{2}}$, multiplied by the rate of either clock $(1-v^2/c^2)^{\frac{1}{2}}$, minus the difference of setting of the clocks due to their separation. If the velocity with which the clock at b was moved from a was W, the difference of setting is

$$\Delta S=\frac{L_s(1-v^2/c^2)^{\frac{1}{2}}}{W}$$

$$\times\left[\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}-\left(1-\frac{(v+W)^2}{c^2}\right)^{\frac{1}{2}}\right]. \quad (8)$$

We must now express this quantity and the product just specified in terms of the observed velocities V_0 and W_0. Using the results of the two preceding papers[1, 2] we get finally for the time as recorded by the clocks on the observing platform:

$$T_p=\frac{L_s}{V_0}\left(\frac{V_0v}{c^2}+\left(1+\frac{V_0^2}{c^2}\right)^{\frac{1}{2}}\right)$$

$$-\frac{L_s}{c}\left(\frac{v}{c}-\frac{1-(1+W_0^2/c^2)^{\frac{1}{2}}}{W_0/c}\right)$$

$$=\frac{L_s}{V_0}\left(1+\frac{V_0^2}{c^2}\right)^{\frac{1}{2}}+\frac{L_s}{W_0}\left(1-\left(1+\frac{W_0^2}{c^2}\right)^{\frac{1}{2}}\right). \quad (9)$$

Let us now compare (9) and (7). We see at once that the transit time is recorded differently by the observed clock and the clocks on the observing platform. Taking certain typical cases we note that:

(I) If W_0 approximates to 0, the transit time is measured *greater* on the observation platform in the ratio

$$\frac{T_p}{T_0}=\frac{(L_s/V_0)(1+V_0^2/c^2)^{\frac{1}{2}}}{L_s/V_0}=\left(1+\frac{V_0^2}{c^2}\right)^{\frac{1}{2}}. \quad (10)$$

This may be stated alternatively that the observed clock appears to run slow as evaluated from the observation platform.

(II) If $W_0=V_0$,

$$T_p=T_0, \quad (11)$$

the transit time is the *same* by either observation.

(III) If W_0 is greater than V_0, the transit time is measured *less* on the observation platform.

By inspection of the equations it is evident that we would have obtained identical answers had we interchanged the observing platform and the body to which the observed clock was attached, that is, had we provided another clock on the observed body, and observed one of the original clocks in its transit between the separated clocks of the new observing platform. This means that we have, in the observations (I), (II), and (III) above, certain consequences of the performance of a sequence of *operations*. The observations do *not* tell us that the "observed" clock is fast or slow with respect to a similar clock on the observation platform. They tell us how the "observed" clock indicates with respect to the difference of indication of clocks transported at various velocities between two points on what we have chosen and labeled as the observation platform.

[2] "The Aberration of Clocks and the Clock Paradox," J. O. S. A. **27**, 305 (1937).

We shall next investigate the question of lengths on the observing platform and on the observed body. To do this we imagine marks or traces made from a and b on the passing observed body, these marks being made at the same clock readings at a and b, that is apparently simultaneously. How far apart will these traces measure on the observed body?

The complete solution of this problem is given only by very complicated formulae, for the reason that, in accordance with Eq. (13) of the preceding paper, the divisions which can be laid off on the observation platform and the observed body are functions of the velocities with which the standard measuring rod and the clocks used in the process are moved past the bodies. These could be different for each body, resulting in apparent contractions or expansions over and above those introduced by the relative motion of the observation platform and observed body, which are our chief concern. We shall here merely note this possibility of a variety of obtainable scale values on the two bodies, and confine our attention to the case that on both our bodies divisions have been marked off by placing a standard measuring rod alongside, that is a rod moving at infinitesimal velocity with respect to the body to be marked. With this simplification, Eqs. (7) and (13) of the previous paper give the solution. In terms of our present symbols we have

$$L_0 = L_s\left[\left(1+\frac{V_0^2}{c^2}\right)^{\frac{1}{2}} + \frac{V_0}{W_0}\left(1-\left(1+\frac{W_0^2}{c^2}\right)^{\frac{1}{2}}\right)\right]. \quad (12)$$

As before we consider special cases:

(I)′ If W_0 approximates to 0, the distance measured on the observed body approximates to $L_s(1+V_0^2/c^2)^{\frac{1}{2}}$, that is the distance is measured *greater* than on the observing platform. An alternative statement is that a measuring rod on the observed body acts as though shorter.

(II)′ If $W_0 = V_0$, we have

$$L_0 = L_s\left[\left(1+\frac{V_0^2}{c^2}\right) + \frac{V_0}{V_0} - \frac{V_0}{V_0}\left(1+\frac{V_0^2}{c^2}\right)^{\frac{1}{2}}\right] = L_s, \quad (13)$$

the distance between traces measures the *same* as the distance between observing stations.

(III)′ If W_0 is greater than V_0, the traces on the observed body are evaluated as separated *less* than the stations on the observation platform.

We therefore find an exactly parallel condition of affairs as for the comparison of indicated times, and the comments made on that case also apply here. The operations we have gone through, starting with the labeling of one body as the observation platform, picking two observing stations on it, and establishing similar clocks at these stations by transport, lead, by use of the method of coincidences, to traced lengths on the observed body which range from apparent contractions to apparent expansions, depending on the observed velocities of transport of the clocks on the observing platform.

From the standpoint of this series of papers all the results (I), (II), (III), (I)′, (II)′, (III)′ are of coordinate validity and significance. They are observations following upon certain operations performed on bodies subject to the Fitzgerald-Larmor-Lorentz contractions in their motion through the ether. Of these possible observations of lengths and indicated times, those designated I and I', corresponding to infinitesimal velocities of the clocks and rods used to establish distances and times at separated stations on the observation platform, are of interest for the reason that they exhibit, upon a certain choice of descriptive terms, a striking *mimicry* of the length and frequency contractions with respect to the ether, which are the underlying causal phenomena. Thus if we use the term "stationary" to describe the observation platform, and describe the observed bodies and clocks as "in motion with respect to" the observation platform (with the reservation that these terms have no real justification, since all that we can observe is relative motion), we find on going through the operations above described that the "moving" bodies and clocks are observed as contracted in length and frequency in the ratio:

$$\frac{1}{(1+V_0^2/c^2)^{\frac{1}{2}}} : 1 = \left(1-\frac{V_0^2}{c^2}+\frac{V_0^4}{c^4}-\cdots\right)^{\frac{1}{2}} : 1.$$

This mimicry, which holds to the second order of V_0/c, becomes exact if in place of V_0 we express the result in terms of v_{2-1} of a preceding paper,[1] for the condition that the clock on the observing platform has been moved infinitely slowly, for

which the symbol v_0, may be used.[3] To do this we put v_2-v, for V; we then have

$$V_0=\frac{v_2-v_1}{(1-v_1^2/c^2)^{\frac{1}{2}}(1-v_2^2/c^2)^{\frac{1}{2}}} \tag{14}$$
$$=\frac{(v_2-v_1)/(1-(v_1v_2/c^2))}{(1-(v_2-v_1)^2/c^2(1-(v_1v_2/c^2)^2)^{\frac{1}{2}}}=\frac{v_0}{(1-v_0^2/c^2)^{\frac{1}{2}}}.$$

Putting this in the expression for L_0 we have

$$L_0=L_s\left(1+\frac{V_0^2}{c^2}\right)^{\frac{1}{2}}$$
$$=L_s\left(1+\frac{v_0^2}{c^2(1-v_0^2/c^2)}\right)^{\frac{1}{2}}=\frac{L_s}{(1-v_0^2/c^2)^{\frac{1}{2}}}, \tag{15}$$

indicating that the measuring rod carried on the observed body appears shortened in the ratio $(1-v_0^2/c^2)^{\frac{1}{2}}:1$. Similarly we find the time interval by the clock on the observed body appears shorter in the same ratio, and hence that the frequency of the observed clock is reduced in that ratio. These apparent contractions thus display a *formal* similarity to those contractions of length and frequency in the ratio $(1-v^2/c^2)^{\frac{1}{2}}:1$ due to motion through the ether from which all the results (I), (I)′, (II), (II)′, (III), (III)′, are derived. Physically this means that any results deduced for motion through the ether will be expressible in formally similar equations in terms of relative motion of matter, *provided* that times and lengths be measured by the sequence of operations leading to I and I', and relative velocities by the convention symbolized by v_0.

The Special Theory of Relativity is, in the light of this analysis, a structure built on *the special case* covered by I and I', and the use of the terms "stationary" and "moving" to describe the observation platform and the observed bodies. By virtue of its actual, though unacknowledged, derivation from the ether and the contractions of length and frequency which occur on motion with respect to the ether, it predicts correctly the results of performing *certain selected operations*. Neglecting to inquire into what physical behaviors underly the validity of this special case, *it offers no explanation of what lies behind the particular choice of operations.*[4] It thus gives only a partial account of the phenomena of moving bodies, in contrast to the complete account[5] given by the hypotheses of Fitzgerald, Larmor and Lorentz, based on the phenomena of aberration and the Michelson-Morley experiment. It has an unquestionable *practical* interest, since the terrestrially attainable velocities of rods and clocks are usually negligible compared with the velocity of light, and it is instinctively easy to regard the earth as "stationary." It has the further element of practicality at the present time that, with the current vogue of radio time signals, its times at distant points coincide with these signals under the simple *convention* that these are always to be assigned the constant value of velocity c. A special case of a general theory should not, however, be allowed to usurp the position properly belonging to that general theory.

[3] V_0 and v_0 are two different measures of the relative velocity of the observation platform and the observed bodies. V_0 is obtained from a clock fixed in position on either platform or body, in conjunction with observation of scale divisions going past; v_0 is obtained from the passage of a fixed point on the observed body past two clocks, one transported infinitely slowly to its station.

[4] The constancy of the measured value of the velocity of light—a feature of the conditions I and I'—is introduced by Einstein in his original communication as "in agreement with experience." This "experience" consists of measurements made with clocks and rods moved at infinitesimal speeds with respect to the bodies under observation. Had "experience" been sufficiently wide, i.e., had it included experiments with rapidly moved clocks and rods, it would (if the contractions are in fact of the character here assumed) have demanded qualification or abandonment of this assumption. The restriction to infinitesimal velocities should be openly included in the statement of the second postulate of the Special Relativity Theory, and when so stated the postulate *loses its appearance of fundamental simplicity* and demands a justification. This lack of any explanation is noticed by Levi-Civita (Nuovo Cimento, **13**, 45–65 (1936)) who asks "if it is possible to give any concrete interpretation . . . to the elementary chronotopic interval." The ether and the length and frequency contractions do give a concrete interpretation.

[5] An outstanding merit of the use of the ether as the reference frame is that it handles without difficulty the positive effects obtained in observations with rotating bodies, such as the Sagnac experiment.

The Dependence of the Kerr Electro-Optic Effect on Temperature, for Carbon Disulphide and Halowax Oil

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(Received July 12, 1937)

INTRODUCTION

THE original aim of this investigation was the determination of the Kerr effect—together with such dielectric constant and refractive index measurements as were needed to check with theory—for a number of liquids, over a wide range of temperature and wave-length. While a considerable amount of work[1] has been done on the temperature variation of the Kerr effect, the agreement with theory (Langevin-Born) is not good. There is also a distinct lack[2] of information as to the influence, if any, of temperature on Kerr dispersion, based on measurements over a large temperature and wave-length interval.

In accordance with this aim a very large series of Kerr measurements was made by the spectro-radiometric method already developed[3] on the materials previously investigated, *viz.*, nitrobenzene, chlorobenzene, *o*-nitrotoluene, carbon disulphide, and halowax oil, over temperature ranges extending nearly from freezing to boiling points and wave-length ranges from the sodium lines to 2μ. In spite of great care in controlling purity and other conditions, however, the determinations on the first three finally turned out to be worth very little, in our estimation. Although small, the current[4] through the cell at the higher temperatures apparently resulted in an inevitable loss of purity which caused a serious variation in the measurements.[5] Accordingly the results as reported here are confined to carbon disulphide —the "standard" liquid for Kerr measurements —and to halowax (chloronaphthalene) which, while they show much smaller Kerr constants, allow measurements which are, in general, reproducible and worthy of confidence.

KERR MEASUREMENTS

I. Carbon disulphide

The Kerr determinations[6] for this material were made in a glass cell with chromium-plated brass plates $8\times50\times188$ mm long, 10 mm apart. The cell was mounted in a double-walled copper container, heavily insulated and of large heat capacity. By evaporating liquid air in varying small amounts in the hollow walls of this container, the cell and contents could be brought down to and kept at any desired low temperature, which, because of the large heat capacity and good insulation, could be held steady to about 1/10 of a degree while measurements were in progress. The cell temperature was measured with a normal pentane thermometer, which had been checked against a thermocouple inserted (when comparing) beside it in the cell. The high thermal conductivity of the container and Kerr plates, as well as the action of a small stirrer, assured uniformity of this temperature.

The carbon disulphide was purified by agitation with mercury and again with activated alumina (for drying) and distilling. It was colorless, almost odorless, and of negligible conductivity. Measurements were made over a range from room temperature down to the freezing point (−111.6°C) for a variety of wave-lengths

[1] For a number of references see J. W. Beams, Rev. Mod. Phys. **4**, 154 (1932); G. Szivessy, *Handbuch der Physik*, Vol. 21 (Julius Springer, 1929), p. 771. See also H. A. Stuart and H. Volkmann, Zeits. f. Physik **83**, 444 (1933) and Ann. d. Physik **18**, 121 (1933); W. Herzog, Zeits. f. Physik **97**, 233 (1935); F. Gabler and P. Sokob, Zeits. f. techn. Physik **17**, 197 (1936).

[2] See Szivessy, reference 1, p. 780.

[3] L. R. Ingersoll and W. R. Winch, Phys. Rev. **44**, 399 (1933).

[4] For a valuable discussion of the methods of purifying nitrobenzene and the relative impossibility of obtaining this material conduction-free, see R. Möller, Physik. Zeits. **32**, 697 (1931).

[5] Stuart and Volkmann's work on the effect of association in liquids on the Kerr effect may also have a bearing here. See Zeits. f. Physik **83**, 444 and 461 (1933). In our case, we were able to obtain very good individual curves but they were not, in general, reproducible. Our experience was such as to lead us to doubt the reliability, in the case of these three liquids, of Kerr measurements extending over a wide temperature range.

[6] For method and apparatus see Ingersoll and Winch, reference 3, p. 400. Since all the measurements here are merely relative and, in the tables, are listed with the value at 20°C as unity, it is immaterial whether we refer to the Kerr effect, i.e., phase difference D (reference 3, p. 399), or to the Kerr constant K.

in the near infrared by the spectroradiometric method,[3] and for sodium light, using a G.E. sodium arc and a quarter-wave plate analyzer. While simpler than the usual Kerr optical system, the phase shifts were large enough so that the error from this source was one of the smallest with which we had to contend.

The use made of the infrared measurements is indicated in Fig. 1. This shows that, to an order of accuracy of about 2 percent, and for these rather large temperature and wave-length intervals, the Kerr dispersion in this material is independent of temperature.

The final temperature dependence measurements shown in Fig. 2 and Table I were limited to the visual determinations with sodium light. A number of factors, including field uniformity considerations and the shorter time during which the liquid had to be under electrical stress, rendered the visual method preferable. While the scattering of points, resulting from fitting together the various curves taken at different voltages, leaves much to be desired, the probable error of the final curve is believed to be of the order of 1 percent. Lyon and Wolfram's[7] results, which extend to −78.5°C, are included for comparison.

II. Halowax oil no. 1007

This material was furnished by the Halowax Corporation of New York. While not a pure substance—it is a mixture of approximately 80 percent alpha-chlor-naphthalene and 20 percent beta-chlor-naphthalene—its good Kerr properties and the fact that it had already been tested in another connection[3] justifies its use. It is a clear colorless liquid, easily dehydrated by agitation with activated alumina and distilling. Its average dipole moment is 1.52×10^{-18} e.s.u. Conductivity is negligible.

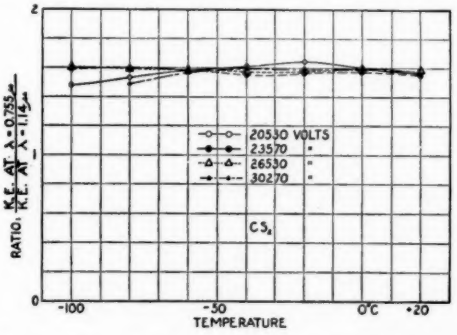

FIG. 1. Curves showing ratios of the Kerr effect in CS_2 at $\lambda=0.755\mu$ and 1.14μ. The fact that all points are close to the mean value (1.58) shows that the Kerr dispersion is independent of temperature.

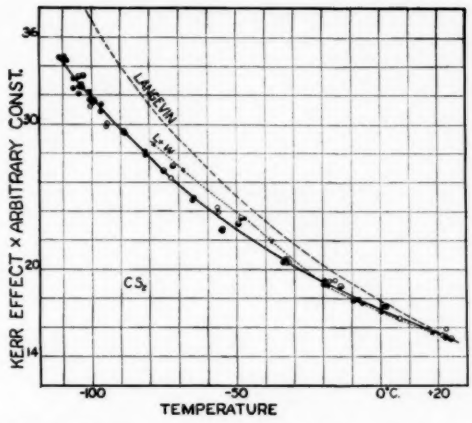

FIG. 2. Temperature variation of the Kerr effect in CS_2. Solid curve is a composite of three sets of determinations made at voltages of 17,400, 23,150, and 26,570 reduced to same scale. Dotted curve, Lyon and Wolfram's results. Dash curve, theoretical values (Langevin).

It was used in an all-Pyrex cell (made to our design by Eck and Krebs) with chromium-plated brass plates 3×47×90 mm, 7 mm apart. Fused quartz[8] separators were used. The cell was mounted in an electrical heating arrangement and the temperatures were read with a mercury thermometer checked against a thermocouple. The results are shown in Fig. 3 and Table II.

Dielectric Constant Measurements

To apply the Langevin-Born theory it is necessary to know the dielectric constant. While this has been measured for CS_2 down to low temperatures,[9] the results are so much at variance at the low temperature end of the curve that it was decided to make independent determinations. For the method and apparatus we are indebted

[7] N. Lyon and F. Wolfram, Ann. d. Physik **63**, 739 (1920).

[8] Those in the CS_2 cell were of glass because, since they fastened near the center of the thick brass plates, a compensating expansion or contraction was required to keep the cell spacing the same at all temperatures.

[9] H. Isnardi, Zeits. f. Physik **9**, 178 (1922); Lyon and Wolfram, reference 7, p. 749.

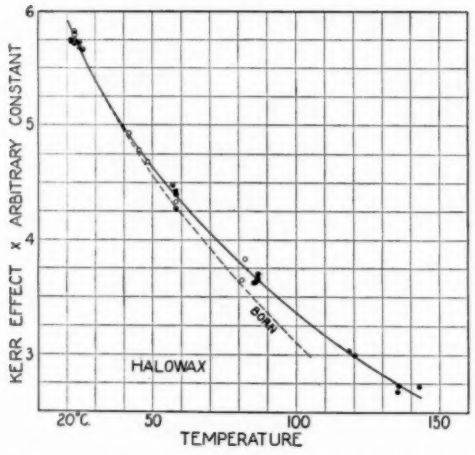

FIG. 3. Temperature variation of the Kerr effect in halowax oil. Curve is a composite of three sets of measurements made at a voltage of 12,500. Dash curve, theoretical values (Born).

to Professor J. W. Williams[10] of the department of chemistry of this university. In this method the capacity change of a variable test condenser is measured by a heterodyne-beat method, when empty and when filled with the liquid, for two settings of the rotor in each case. The dielectric constant is obtained at once as the ratio of the two capacities. Considerable thought was given to the design of the test condenser so that the inevitable change of dimensions with temperature should not cause a displacement or looseness of the rotor and hence uncertainty in measurement. The condenser finally used was of a cylindrical type[11] employing a vertical rotor with widely separated conical bearings. The temperature control arrangements were somewhat similar to those used with the Kerr cell. The results are shown in the tables, and, for CS_2, in Fig. 4. It will be noted that our determinations are in good agreement with the earlier observations down to $-50°C$, below which point there is considerable divergence. Our curve, particularly the part with greater slope below $-92°C$, was checked several times with two different condensers. Measurements also of density and refractive index (with a Zeiss Pulfrich refractometer) were carried out for halowax, to allow theoretical calculations.

TABLE I. *Carbon disulphide.*

Temp. °C	ϵ	d	n	Kerr Constant obs.	Kerr Constant calc. (Langevin)
20	2.626	1.263	1.628	1.000	1.000
10	2.657	1.278	1.635	1.047	1.071
0	2.688	1.293	1.643	1.103	1.144
− 10	2.716	1.307	1.651	1.161	1.224
− 20	2.740	1.321	1.659	1.227	1.305
− 30	2.765	1.335	1.667	1.295	1.396
− 40	2.792	1.350	1.675	1.373	1.495
− 50	2.814	1.364	1.683	1.458	1.606
− 60	2.839	1.378	1.691	1.546	1.730
− 70	2.863	1.392	1.699	1.655	1.859
− 80	2.885	1.407	1.707	1.775	2.010
− 90	2.910	1.421	1.715	1.900	2.180
−100	2.954	1.435	1.723	2.041	2.381
−110	2.988	1.450	1.731	2.197	2.610

Notes: The dielectric constant ϵ is that determined in the present work. The density d is from I.C.T., Vol. 3, p. 23. Refractive index n from Lyon and Wolfram (reference 7), the last 5 values being extrapolated.

[10] See Daniels, Mathews, and Williams, *Experimental Physical Chemistry* (McGraw-Hill, 1934), p. 255.

THEORETICAL CONSIDERATIONS

The Born-Langevin theory gives for the dependence of the Kerr effect on temperature the expression[12]

$$B = C\frac{(\epsilon+2)^2(n^2+2)^2 d}{nT}\left(1+\frac{A}{T}\right),$$

where ϵ is the dielectric constant, n refractive index, d density, T absolute temperature, and C and A quantities independent of temperature, the last involving the dipole moment of

TABLE II. *Halowax oil.*

Temp. °C	ϵ	d	n	Kerr Constant obs.	Kerr Constant calc. (Langevin)	Kerr Constant calc. (Born)
20	5.73	1.229	1.635	1.000	1.000	1.000
30	5.59	1.217	1.631	0.916		
40	5.45	1.205	1.627	0.845	0.846	0.835
50	5.34	1.194	1.623	0.785		
60	5.22	1.183	1.619	0.733	0.730	0.713
70	5.13	1.172	1.615	0.688		
80	5.04	1.162	1.610	0.645	0.637	0.616
90	4.93*	1.151	1.606	0.605		
100	4.84*	1.141	1.602	0.569	0.548	0.525
110				0.535		
120				0.507		
130				0.478		
140				0.455		

* Extrapolated.

[11] Modified from a design by P. S. Albright, which will be described by him in a forthcoming article.

[12] Szivessy, reference 1, p. 770.

the molecule. The value of A may be taken[13] as approximately equal to $M^2/K\beta$ where M is the dipole moment, K the Boltzmann constant, and $\beta = 3/2$ mol. wt./density $\times 6 \times 10^{23}$. In the case of halowax the Born correction term A/T has a value of the order of one-tenth. A is, of course, zero for CS_2, so the simple Langevin formula applies here.

The dash lines of Figs. 2 and 3 are calculated on the basis of this theory. It will be noted that in each case the theoretical curve is steeper than the observed, and this has, in general, been found to be the case by other observers. As Table II shows, the simple Langevin formula, omitting dipole moment considerations, actually fits the case better than the other. In the matter of the dependence of dispersion on temperature, theory[14] is not entirely satisfactory, but indicates in general that the dispersion should be independent of temperature (save for a small second-order effect connected with the variation of refractive index with temperature). This, we have seen, is the case for CS_2. Szivessy[14] has shown, for a much smaller wave-length and temperature interval, that the same is true for nitrobenzene.

Conclusions

1. The Kerr dispersion of carbon disulphide is independent of temperature for rather wide temperature and wave-length ranges. The fact that our measurements on a nonpolar liquid and Szivessy's on a highly polar one (the latter, however, over only a very restricted wave-length and temperature interval) both agree, in general, with theoretical predictions, indicates that this temperature independence of Kerr dispersion is very likely generally true.

2. The Langevin-Born theory does not represent well the temperature variation of the Kerr effect either for nonpolar carbon disulphide at low temperatures or polar halowax at high ones.

3. The dielectric constant *vs.* temperature curve for carbon disulphide shows an abrupt increase of slope at $-92°$C.

We wish to acknowledge our indebtedness to the Wisconsin Alumni Research Foundation for its aid in supporting this research, and we also wish to thank Messrs. Kenneth McDougal and Wesley R. Winch for assistance in many of the earlier measurements.

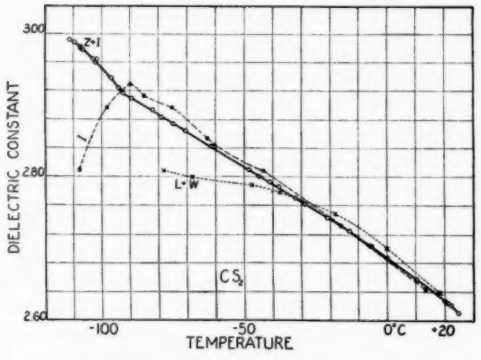

FIG. 4. Temperature variation of dielectric constant in carbon disulphide. Isnardi's and Lyon and Wolfram's results are shown for comparison.

[13] Reference 7, pp. 741, 745.
[14] See Szivessy, reference 1, p. 780.

The Diffraction Pattern Due to an Absorbing Prism

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An equation is derived for the case of Fraunhofer diffraction through an absorbing prism limited by an aperture. This reduces for the case of zero absorption to the usual equation, and for the case of an infinite absorbing prism, to a witch. Curves are plotted to show how the diffraction bands fade out with increased absorption. The theory is checked by photographs of light sent through a liquid prism of India ink solution. The chromatic resolving power of an infinite absorbing prism is derived. The equation is modified for the grazing angles used in x-rays and an angular half-width of the diffraction pattern of 2.37″ of arc is computed for the case of Bearden and Shaw's experiment on the refractive index of quartz at 1.54A.

I. INTRODUCTION

THE effect of absorption in optical prisms may usually be neglected except when they are used for the infrared or extreme ultraviolet. In the field of x-rays, however, absorption is usually an important factor. For example, Bearden and Shaw[1] have measured the index of refraction for the characteristic x-rays from copper, passing through the edge of a 90° quartz prism. They stated that the absorption in the quartz limited the effective width of the beam to not more than 0.001 mm. It was this statement which led the author to investigate the diffraction pattern due to an absorbing prism, a preliminary report of which has been presented.[2]

II. THE DIFFRACTION FORMULA

We shall consider first an absorbing prism so thin that the beam is not deviated appreciably by refraction. A plane wave passes normally through the prism, which is limited by a rectangular aperture of width a and height b. (See Fig. 1.) One corner of the aperture, O is the origin of a system of rectangular coordinates

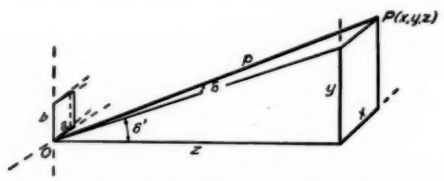

FIG. 1. Absorbing prism, axb and point P on distant screen.

with the x and y axes in the direction of a and b. The prism edge coincides with the y axis.

The problem is to derive an expression for the amplitude at a point P on a distant screen. The derivation will follow the usual one for Fraunhofer diffraction of light through a narrow aperture except for the inclusion of the amplitude absorption factor e^{-ka}. As usual we shall assume the dimensions of the aperture to be small compared to all other distances.

The coordinates of P are

$$\begin{aligned} x &= p \sin \delta, \\ y &= p \sin \delta', \\ z &= p \cos \delta \cos \delta'. \end{aligned} \tag{1}$$

The contribution of an element of area $dadb$ to the amplitude A at P will have a phase angle $\phi+\phi'$ where

$$\phi = (2\pi/\lambda)a \sin \delta = \sigma a = 2\beta \tag{2}$$

and ϕ' is an analogous expression in δ'.

$$dA = (j/\lambda p)e^{-(k+j\sigma)a}e^{-j\sigma' b}dadb, \tag{3}$$

where $j=(-1)^{\frac{1}{2}}$. When (3) is integrated over the area of the aperture and then multiplied by its complex conjugate, the result is the intensity.

$$I = \frac{b^2}{\lambda^2 p^2} \frac{(1+e^{-2ka}-2e^{-ka}\cos \sigma a)}{(k^2+\sigma^2)} \frac{\sin^2 \beta}{\beta^2}. \tag{4}$$

In case the absorption is zero (4) reduces to the usual expression for Fraunhofer diffraction through a rectangular aperture. When the absorption is so large that the amplitude factor e^{-ka} for the far edge of the aperture is negligible,

[1] J. A. Bearden and C. H. Shaw, Phys. Rev. **46**, 759 (1934).
[2] R. C. Spencer, Phys. Rev. **48**, 478A (1935).

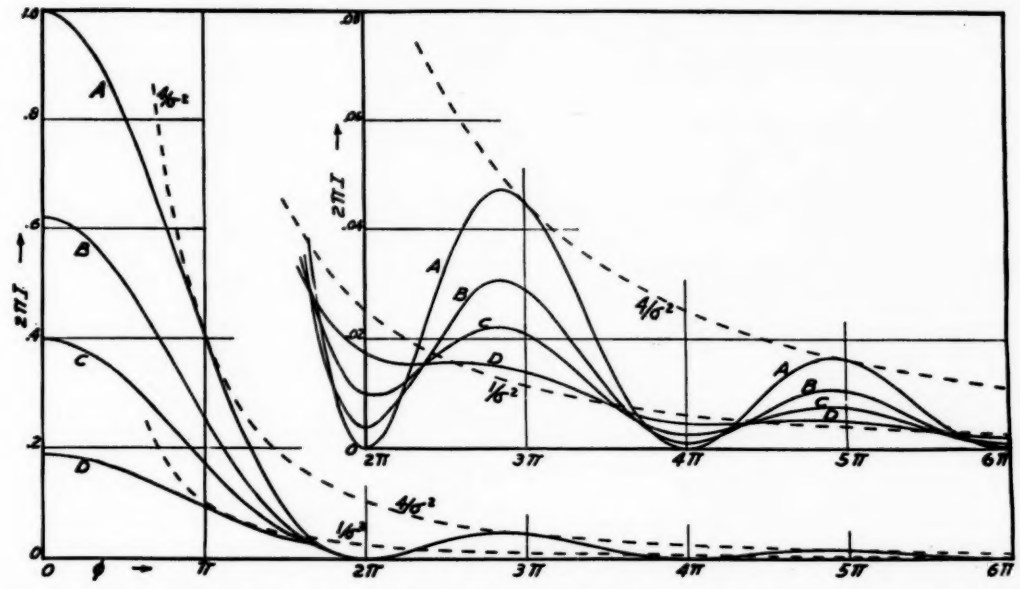

FIG. 2. Curves calculated from the diffraction formula for various values of absorption. Diffraction formula for absorbing aperture, $2\pi I=\frac{1+e^{-2ka}-2e^{-ka}\cos\sigma a}{k^2+\sigma^2}$. Curve ka: $A=0$, $B=\frac{1}{2}$, $C=1$, $D=2$.

(4) reduces to

$$I=\frac{b^2}{\lambda^2 p^2}\frac{1}{(k^2+\sigma^2)}\frac{\sin^2\beta}{\beta^2}. \tag{5}$$

The factor $1/(k^2+\sigma^2)$ is a witch with a half-width at half-maximum of $\sigma=k$. Assuming that $\sin\delta=\delta$, the *angular half-width* at *half-maximum* is

$$d=\lambda k/2\pi, \tag{6}$$

which is the *amplitude absorption coefficient per* $1/2\pi$ *wave-lengths.*

Description of diffraction pattern

Henceforth we will consider the properties of the horizontal pattern only. Fig. 2 shows a family of curves calculated from the equation

$$2\pi I=(1+e^{-2ka}-2e^{-ka}\cos\sigma a)/(k^2+\sigma^2). \tag{7}$$

The four oscillating curves, A to D (shown magnified tenfold in the insert), are plotted for values of $ka=0$, $1/2$, 1 and 2. Curve A is the usual one for Fraunhofer diffraction through a narrow aperture. The dashed curve $4/\sigma^2$ is tangent to the crests of the zero absorption curve. The dashed curve $1/\sigma^2$ is asymptotic to the curves for large values of ka.

The oscillations fade out rapidly with increased absorption. From (7) we find that the maximum fractional deviation from the mean value is *sech ka*, which represents the clearness of the band structure against the background. At large angles the mean value of $2\pi I$ lies between $2/\sigma^2$ and $1/\sigma^2$ regardless of width of aperture or absorption coefficient.

Equivalent width

The problem is to determine the width a_0, of a nonabsorbing aperture, which will have a diffraction pattern equivalent in some respect to that of an absorbing aperture a. There are a number of properties on which to base such a comparison, and each leads to a different expression for a_0. Three of these properties, are discussed below. The subscript, zero, refers to the case of zero absorption.

1. *Equal flux*

$$a_0=(1-e^{-2ka})/2k \tag{8}$$

gives the width of the nonabsorbing aperture

which will transmit the same total flux as the absorbing aperture a.

2. *Equal half-width at half-maximum*

$$a_0 = 2.7831/k. \tag{9}$$

This comparison is for the case of complete absorption.

3. *Equal resolving power.*—In the case of zero absorption, Lord Rayleigh assumed that the angle between two equally intense monochromatic spectral lines may be resolved when the maximum of one diffraction pattern falls on the first minimum of the other. The angular separation λ/a_0 is called the angular resolution, or resolving power of the aperture. The resultant of the two patterns contains a minimum at the center which is $8/\pi^2$ or 0.810 of either side maximum.

Let us assume in the case of complete absorption that two equally intense monochromatic spectral lines may be resolved when their separation is equal to the full width at half-maximum of the diffraction pattern. Expressing (5) by $1/(1+x^2)$, the resultant pattern formed by the two witches is given by

$$y = \frac{1}{1+(x+1)^2} + \frac{1}{1+(x-1)^2} = \frac{2(2+x^2)}{4+x^4}. \tag{10}$$

There exists a central minimum between two maxima, the positions of which are obtained from the positive root of

$$x^4 + 4x^2 - 4 = 0, \tag{11}$$

which is $x^2 = 2\sqrt{2} - 2$. The ratio of central minimum to maximum is $2\sqrt{2} - 2$ or 0.828, which agrees very well with the value of 0.810 for the corresponding case of zero absorption. Therefore, we may assume that the two definitions are equivalent.

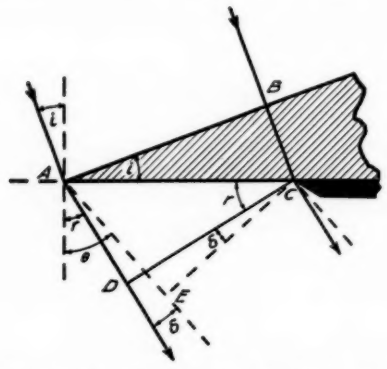

FIG. 3. Diffraction through an absorbing prism.

The angular separation of the two witches is $2d$. From (6)

$$2d = \lambda k/\pi. \tag{12}$$

Equating (12) to λ/a_0 results in

$$a_0 = \pi/k. \tag{13}$$

III. APPLICATION TO LIGHT

Modification of formula for prisms

When prism spectroscopes are used in the infrared, or ultraviolet, the problem of absorption becomes important. In Fig. 3 the wave front AB is incident normally upon one face of the half-prism BAC. It then passes through the second face at an angle of incidence i and an angle of refraction r. A stop at C limits the beam to a width of

$$a = DC = AC \cos r.$$

Absorption in the prism material reduces the intensity at C by the factor $e^{-\mu t}$ where $t = BC$ is the distance the light has traveled through the prism and μ is the linear absorption coefficient. The integration is carried out over the prism face AC to derive the amplitude of the diffraction pattern in a direction making an angle $\theta = r + \delta$ with the normal. Eqs. (2)–(7) are again found to hold since

$$k = \frac{\mu t}{2a} = \frac{\mu}{2} \frac{\sin i}{\cos r} \tag{14}$$

and

$$(\sin\theta - \sin r)/\cos r$$

may be approximated by δ.

Resolving power of infinite absorbing prism

If, in the derivation of the resolving power of a prism, we use the angular resolution for an infinite absorbing aperture (12) in place of λ/a, for a nonabsorbing aperture, we obtain the chromatic resolving power

$$\lambda/\Delta\lambda = (2\pi/\mu)(dn/d\lambda), \tag{15}$$

where n is the index of refraction of the prism. The resolving power is the same as that of a

nonabsorbing prism with a base

$$t_0 = 2\pi/\mu. \tag{16}$$

Photographic check

Suitable absorbing glass was not at hand with which to construct a prism. Therefore, the type of liquid absorbing prism shown in Fig. 4 was

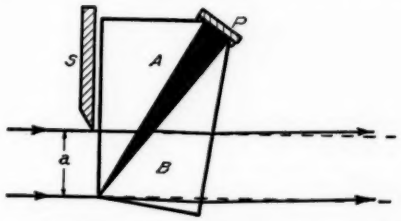

FIG. 4. Experimental absorbing prism using a solution of India ink between two glass prisms A and B.

adopted. Two 30° glass prisms, A and B, and a plate P, were waxed to a base plate to form a hollow prism with an angle of $5\frac{1}{2}°$. This was filled with a solution of India ink in water, and then covered to prevent evaporation. It was placed on the prism table of an ordinary spectrometer.

A Point-O-Light lamp with condensing lens and green filter was used to illuminate the collimator slit. The diffraction pattern was photographed by pulling out the eyepiece until the pattern was focused on the photographic film placed 20 cm back of the eyepiece. The exposure time for Fig. 5(a) was 10 seconds, and varied from 10 to 20 minutes for Figs. 5(b) to 5(f).

Figure 5 shows the photographs obtained. Clear water was used in Fig. 5(a). The aperture of 1.6 mm gave sharp diffraction bands. In Fig. 5(b) a 6 percent solution of India ink in water was used with an 8 mm (practically infinite) aperture. The bands have disappeared, leaving a diffused central portion which is the witch indicated in (5). The absorption was increased in Fig. 5(c) by using a 10 percent solution and by turning the prism clockwise 25° to lengthen the path through the liquid. The resultant widening of the diffraction pattern is predicted by theory.

Figures 5(d), (e) and (f) were taken with an 8 percent solution, 12° rotation, and with apertures of 0.7, 0.5, and 0.3 mm. No diffraction bands were distinguished for apertures greater than 0.7 mm, but starting with that width, they become clearer as the aperture is made narrower. This is in qualitative agreement with the theory.

IV. Application to X-Rays

Modification of formula at grazing angles

We will limit the discussion to the case of complete absorption. It is customary in x-ray work to measure the grazing angle between the ray and the surface, rather than the angle between the ray and the normal. Thus, the sine and cosine terms are interchanged and (7) becomes

$$2\pi I = 1/(k^2+\sigma^2). \tag{17}$$

The angular half-width (6) at grazing angles becomes

$$d = \lambda\mu/4\pi r. \tag{18}$$

Calculations

The following calculations were made for the conditions of Bearden and Shaw's experiment[1] in which they obtained the absolute x-ray wavelength of the Cu $K\alpha$ doublet from measurements of the index of refraction in quartz. Referring to their Table I, we will use values of angles of incidence and deviation, $\alpha = 30'$ and $\beta = 3'$ of arc. The value of the angle of refraction r, for the method of internal incidence, is the sum of the two, $r = 33'$ of arc.

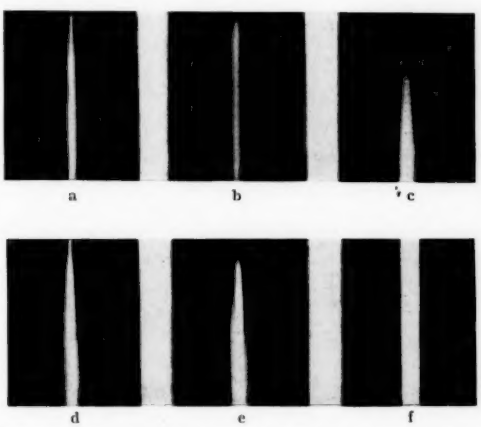

FIG. 5. Photographs of light diffracted through liquid prisms of solutions of India ink and water: (a) for zero absorption shows sharp diffraction bands; (b) and (c) for complete absorption indicate the witch of Eq. (5). The bands gradually appear in (d) to (f) as the aperture is decreased.

The linear absorption coefficient μ, was computed from tables[3] and found to be 90.17 per cm at 1.54A.

1. *Equivalent aperture.*—The width of a non-absorbing aperture, a_0, which would transmit the same energy flux as the quartz prism is found from (8) to equal 0.00106 mm. This agrees with the statement made by Bearden and Shaw, "The absorption in quartz is such that for the Cu $K\alpha$ lines the effective depth of prism used is not more than 0.001 mm." Attention is called to the other types of equivalent apertures (9) and (13).

2. *Angular half-width.*—The value of the angular half-width obtained from (18) is $d = 2.37''$ of arc. The full width at half-maximum is $4.75''$. This is nearly half of the value of $9.8''$ given as the rocking curve width for the calcite crystals in the (1–1) position, and nearly a quarter of the full width of the Cu $K\alpha_1$ component. (See their Fig. 4.)

3. *Increase in width of doublet.*—It is well known that if an x-ray line obeys the Hoyt equation,[4] which has the shape of a witch, and if the diffraction pattern also has the shape of a witch, then the resulting experimental curve is a witch with a width equal to the sum of the two widths.[5] Therefore, the increase in half-width of the Cu $K\alpha_1$ component should be $2.37''$ of arc. An inspection of their Fig. 4 shows that the increase in full width at half-maximum of the unresolved doublet is insensitive to the change in width of the smaller component and is approximated by the increase in half-width of the larger component which is $2.37''$ of arc. This agrees with the statement by Bearden and Shaw that, "The refracted beam was about $2''$ wider than one would expect from the different angles of incidence of the two sides of the line."

[3] Compton and Allison, *X-Rays in Theory and Practice* (Van Nostrand), p. 802.
[4] Archer Hoyt, Phys. Rev. **40**, 477 (1932).
[5] F. K. Richtmyer, S. W. Barnes and E. Ramberg, Phys. Rev. **46**, 843 (1934), Eq. (3).

V. Conclusion

The new diffraction formula, which considers the effect of absorption in prism materials, shows a gradual transition from the usual Fraunhofer diffraction bands obtained when the absorption is zero, to the smooth witch obtained when the absorption is large. The formula should serve as a guide in the design of prism spectroscopes for wave-length regions of high absorption. The tendency for the absorption diffraction pattern to widen spectral lines should be considered in the study of line widths and shapes.

The theory has been applied to the calculation of the resolving power of an absorbing prism, and was also used to calculate the increase in line widths in x-ray refraction. It is suggested that sound waves might be transmitted through prisms made up of absorbing materials. The resulting diffraction patterns would be strongly influenced by absorption and might be a means of calculating the absorption coefficient through (6) and (14).

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